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On the analogy between gravity modes and inertial modes in spherical geometry

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Abstract. Analysing similarities and differences between gravity modes and inertial modes, we derive the analytic expression of gravity modes in an axisymmetric ellipsoid for a diffusionless fluid. We also give the expression of the dispersion relation for these modes in such a geometry. The case of a spherical shell is then considered. After deriving a class of analytical solutions, we exhibit the analogous structure of singular gravity modes and singular inertial modes. Finally, we show why singular modes should play an important part in the transport properties of a stratified or rotating fluid.

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1 Introduction

Gravity waves owe their existence to buoyancy force while inertial waves use Coriolis force as the restoring force. The dispersion relation of such waves is very similar and shows that the frequency depends only on the angle between the wave vector and the gravity or rotation axis, a peculiarity which implies that phase and energy propagate perpendicularly to each other. The similarity between these waves leads to a number of interesting results which have been first reviewed by Veronis [1]. Here we shall focus on the fact that pressure perturbations obey the same partial differential equation namely Poincaré equation. Therefore when one considers the proper oscillations of a stably stratified fluid or a rotating fluid contained in a bounded volume, one faces the same mathematical problem.

This problem has the distinctive feature that it is mathematically ill-posed in the sense of Hadamard: Poincaré equation is indeed hyperbolic with respect to spatial variables while boundary conditions to be met are of Dirichlet type (in fact with oblique derivatives). The search of proper oscillations therefore leads to a rather difficult eigenvalue problem which has aroused renewed interest very recently with the works of Maas and Lam [2] for gravity modes, Rieutord and Valdettaro [3] for inertial modes and Dintrans *et al.* [4] for gravito-inertial modes. The results of these works, focused on essentially two-dimensional problems, have shown that solutions of this eigenvalue problem may be classified in three types: a first type consists of regular $\mathcal{C}^\infty$ solutions which are associated with characteristics trajectories following quasi-periodic orbits. For such orbits, characteristics cover the whole hyperbolic domain and the Lyapunov exponent associated with the characteristics paths is strictly zero. They are associated with a discrete eigenvalue spectrum likely dense. A second type of solutions consists of singular solutions associated with periodic orbits of characteristics. For such modes, characteristics converge towards a limit cycle and the Lyapunov exponent is negative. The solution is said to be singular as the velocity field is not square-integrable and the point spectrum of the operator is empty (no eigenvalue exist). In fact the velocity field diverges as the limit cycle is approached. The third type is also made of singular solutions which are associated with characteristics trajectories which are trapped in a wedge formed by either the boundaries of the domain (*e.g.* [5]) or a turning surface (which happens when the operator is of mixed type as in [4]).

The above results apply of course to the inviscid problem, however when dissipative processes are taken into account singular and regular modes are much different, essentially because singular modes contain very small scales controlled by viscosity (or thermal diffusion) and are much more damped than regular modes.

The coexistence of singular and regular modes in the spectrum of oscillations of rotating or stratified fluids has important consequences as far as transport by waves is concerned. This kind of transport is invoked in many fields of geophysics or astrophysics. For instance, in the radiative zone of stars, which are stably stratified, waves are often invoked to explain the transport of angular momentum or chemical elements [6].

Transport properties are however nonlinear properties which are even more difficult to apprehend and resorting to experiments is often a good way to deal with these

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difficult issues. However, rotating or stratified fluids are not so easy to manipulate and according to which range of parameters one wishes to access, it will be easier to use a stratified fluid rather than a rotating one or *vice versa*, provided we make use of the above mentioned analogy.

The aim of the present paper is to show the similarities and differences between inertial and gravity modes in containers which geometry is spherical or close to it. A first account on the transport properties will also be given. We shall therefore set out the equations of the diffusionless problem (Sect. 1) and show how to solve them in the case of a full axisymmetric ellipsoid (Sect. 2). We then turn to the case with diffusion; we further consider the difference between the two types of modes. Solving numerically this case for the spherical shell, we show some examples of singular modes (Sect. 3). As a final discussion we briefly consider the transport properties of such modes.

2 Inertial and gravity modes in a spherical or ellipsoidal container

Let us first consider a spherical shell of outer radius R and inner radius ηR. The fluid inside is assumed inviscid and either rotating at an angular velocity Ω around the z-axis or stably stratified in a gravity field $\mathbf{g} = -g\mathbf{e}_z$.

2.1 Inertial modes

In the case of a rotating fluid, perturbations obey the following linearized equations

$$\mathrm{i}\omega\mathbf{u} + \mathbf{e}_z \times \mathbf{u} = -\boldsymbol{\nabla} p$$
$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0$$

which have been put in adimensional form using $(2\Omega)^{-1}$ as the time scale. $\mathbf{u}$ is the velocity field and p the reduced pressure; time dependence of solutions has been assumed to be as $\exp(\mathrm{i}\omega t)$. Using cylindrical coordinates (s, φ, z), these equations give the expression of the velocity field as a function of the pressure gradient:

$$\begin{cases} u_s = -\dfrac{1}{1-\omega^2}\left(\mathrm{i}\omega\dfrac{\partial P}{\partial s} + \dfrac{1}{s}\dfrac{\partial P}{\partial \varphi}\right), \\ u_\phi = \dfrac{1}{1-\omega^2}\left(\dfrac{\partial P}{\partial s} - \dfrac{\mathrm{i}\omega}{s}\dfrac{\partial P}{\partial \varphi}\right), \\ u_z = -\dfrac{1}{\mathrm{i}\omega}\dfrac{\partial P}{\partial z} \end{cases} \quad (1)$$

while using mass conservation yields Poincaré equation:

$$\Delta p - \frac{1}{\omega^2}\frac{\partial^2 p}{\partial z^2} = 0. \quad (2)$$

The boundary condition $\mathbf{u}\cdot\mathbf{e}_\mathrm{r} = 0$ ($\mathbf{e}_\mathrm{r}$ is the unit radial vector) on the spherical shells gives birth to the oblique derivative conditions for the pressure:

$$s\frac{\partial P}{\partial s} + \frac{m}{\omega}P - \frac{1-\omega^2}{\omega^2}z\frac{\partial P}{\partial z} = 0$$

where it has been assumed a φ-dependence of the form of $\exp(\mathrm{i}m\varphi)$.

2.2 Gravity modes

The spherical shell is now filled with a quasi-incompressible stratified fluid and gravity $\mathbf{g}$ is along the z-axis. We shall assume that the stratification is the result of a uniform temperature gradient but this may be the result of a concentration gradient (which is more convenient for experiments actually). We therefore write

$$\boldsymbol{\nabla} T_\mathrm{eq} = \beta\mathbf{e}_z$$

where β is a positive constant. We also assume a linear relation between temperature and density fluctuations and write

$$\frac{\delta\rho}{\rho} = -\alpha\delta T$$

where α is the (constant) coefficient of volume expansion.

With these quantities we build the Brunt-Väisälä frequency of the system $N = \sqrt{\alpha\beta g}$ which we use to define the time scale.

Non-dimensional linearized equations of infinitesimal perturbations using Boussinesq approximation, read

$$\begin{cases} \mathrm{i}\omega\mathbf{u} = -\boldsymbol{\nabla} p + \theta\mathbf{e}_z \\ \mathrm{i}\omega\theta + u_z = 0 \\ \boldsymbol{\nabla}\cdot\mathbf{u} = 0. \end{cases} \quad (3)$$

These equations may also be transformed into a single one for pressure which is the Poincaré equation:

$$\Delta p - \frac{1}{1-\omega^2}\frac{\partial^2 p}{\partial z^2} = 0 \quad (4)$$

with the relation velocity-pressure:

$$\begin{cases} u_s = \dfrac{\mathrm{i}}{\omega}\dfrac{\partial P}{\partial s} \\ u_\phi = \dfrac{\mathrm{i}}{\omega}\dfrac{1}{s}\dfrac{\partial P}{\partial \varphi} \\ u_z = -\dfrac{\mathrm{i}\omega}{1-\omega^2}\dfrac{\partial P}{\partial z}. \end{cases} \quad (5)$$

The boundary condition $\mathbf{u}\cdot\mathbf{e}_\mathrm{r} = 0$ is now

$$s\frac{\partial P}{\partial s} - \frac{\omega^2}{1-\omega^2}\frac{\partial P}{\partial z} = 0.$$

2.3 The analogy between modes

The above equations clearly show that for axisymmetric modes there is a one-to-one correspondence between gravity modes and inertial modes. It suffices to make the correspondence between ω and $\sqrt{1-\omega^2}$. The velocity field is different however since a pressure fluctuation of an inertial mode induces a toroidal field (v_ϕ) while it does not if it is a gravity mode.

Non-axisymmetric modes however are not equivalent since the boundary conditions are not the same; this implies a different set of eigenvalues.

The similar structure of the two problems allows one to transpose technics developed on one problem to the other. We shall now illustrate this point with the solutions in an axisymmetric ellipsoid.

2.4 Analytic solutions in an axisymmetric ellipsoid

In 1889, Bryan [7] investigated the rotational stability of a self-gravitating MacLaurin ellipsoid; his technique was used by Greenspan to derive the analytical solutions for inertial modes in a full sphere and the associated dispersion relation. Using the above mentioned analogy, it is possible to derive the same results for gravity modes in a full sphere and, in addition, generalize a little the results to the axisymmetric ellipsoid[1].

The first step is to transform Poincaré equation (4) into Laplace equation by making the change of variable:

$$z' = \frac{\sqrt{1-\omega^2}}{\omega}\mathrm{i}z.$$

This transformation however changes the ellipsoid into a one-sheet hyperboloid of equation:

$$\frac{x^2+y^2}{R^2} - \frac{\omega^2}{1-\omega^2}\frac{z'^2}{R_\mathrm{p}^2} = 1$$

where R and R_p are respectively the equatorial and polar radius of the ellipsoid. Using the equatorial radius as the length scale and introducing the flatness $\varepsilon = (R - R_\mathrm{p})/R$ of the ellipsoid, we write the hyperboloid equation as

$$s^2 - \frac{\omega^2}{1-\omega^2}\frac{z'^2}{(1-\varepsilon)^2} = 1. \qquad (6)$$

We need to solve now Laplace equation in this geometry. For this purpose we use ellipsoidal coordinates (χ, ξ, ϕ) which are related to Cartesian coordinates by

$$\begin{cases} x &= a\cosh\xi\cos\chi\cos\phi \\ y &= a\cosh\xi\cos\chi\sin\phi \\ z' &= a\sinh\xi\sin\chi. \end{cases} \qquad (7)$$

The surface of coordinate $\chi = \mathrm{Cte}$ is an axisymmetric hyperboloid of equation:

$$\frac{x^2+y^2}{a^2\cos^2\chi} - \frac{z'^2}{a^2\sin^2\chi} = 1.$$

If we wish to use this hyperboloid as the bounding surface on which boundary conditions are taken, we need taking

$$a^2 = 1 + \frac{(1-\varepsilon)^2(1-\omega^2)}{\omega^2}\cdot \qquad (8)$$

Now we note that

$$z = \mathrm{i}\frac{\omega}{\sqrt{1-\omega^2}}a\sinh\xi\sin\chi$$

therefore, following Greenspan [8], we may simplify the notations by posing

$$\mathrm{i}\sinh\xi = \mu \quad \text{and} \quad a\sin\chi = \eta$$

in which case one has

$$\begin{cases} s = \sqrt{x^2+y^2} = \sqrt{(a^2-\eta^2)(1-\mu^2)} \\ z = \dfrac{1-\varepsilon}{\sqrt{a^2-1}}\mu\eta. \end{cases} \qquad (9)$$

The solutions of Laplace equations may now be written:

$$P = \sum_{l,m} A_{l,m} P_l^m\left(\frac{\eta}{a}\right) P_l^m(\mu)\mathrm{e}^{\mathrm{i}m\phi} \qquad (10)$$

where P_l^m are Legendre polynomials. We refer the reader to Angot [9] for more details on solving Laplace equation for this type of coordinates.

Now using the equations of motion, we note that the velocity field expresses as

$$u_s = \frac{\mathrm{i}}{\omega}\frac{\partial P}{\partial s}, \quad u_\phi = \frac{\mathrm{i}}{\omega}\frac{1}{s}\frac{\partial P}{\partial\varphi}, \quad u_z = -\frac{\mathrm{i}\omega}{1-\omega^2}\frac{\partial P}{\partial z}\cdot$$

Therefore the boundary condition $\mathbf{u}\cdot\mathbf{n} = 0$ on the ellipsoid reads

$$s\frac{\partial P}{\partial s} - \frac{1}{a^2-1}z\frac{\partial P}{\partial z} = 0 \qquad (11)$$

which needs to be rewritten with the (η, μ)-system. Noting that

$$\begin{cases} \dfrac{\partial\mu}{\partial s} = \dfrac{s\mu}{\varDelta}, \quad \dfrac{\partial\mu}{\partial z} &= \dfrac{(1-\mu^2)\eta\sqrt{a^2-1}}{\varDelta(1-\varepsilon)} \\ &= \dfrac{(1-\mu^2)\eta^2\mu}{\varDelta z} \\ \dfrac{\partial\eta}{\partial s} = -\dfrac{s\eta}{\varDelta}, \quad \dfrac{\partial\eta}{\partial z} &= \dfrac{\mu(\eta^2-a^2)\sqrt{a^2-1}}{\varDelta(1-\varepsilon)} \\ &= \dfrac{(\eta^2-a^2)\mu^2\eta}{\varDelta z} \end{cases} \qquad (12)$$

with $\varDelta = \eta^2 - a^2\mu^2$, we have

$$\left(1 - \frac{\eta^2}{a^2-1}\right)a^2(1-\mu^2)\mu\frac{\partial P}{\partial\mu} + \frac{m}{\omega}(\eta^2 - a^2\mu^2)P = \left(\frac{\mu^2}{a^2-1} - 1 + \mu^2\right)(\eta^2-a^2)\eta\frac{\partial P}{\partial\eta}$$

on the surface of the ellipsoid which is given by $\eta = \sqrt{a^2-1}$ with these coordinates.

The dispersion relation finally reads

$$(P_l^m)'\left(\frac{(1-\varepsilon)\sqrt{1-\omega^2}}{\sqrt{\omega^2+(1-\varepsilon)^2(1-\omega^2)}}\right) = 0 \qquad (13)$$

where the prime denotes the derivative. For the sphere $(\varepsilon = 0)$, this formula simplifies into

$$P_l^{m\,\prime}(\sqrt{1-\omega^2}) = 0. \qquad (14)$$

Formulae (13, 14) show that the spectrum of gravity modes in a full ellipsoid (or sphere) is discrete and dense in $[-1,1]$ and the eigenfunctions, given by (10), are $\mathcal{C}^\infty$.

[1] This shape is important to estimate the effects of centrifugal flattening of a star or a planet or, more prosaically, of a spherical container in an experiment.

In the case of inertial modes the equivalent dispersion relation is found to be

$$mP_l^m\left(\frac{(1-\varepsilon)\omega}{S(\omega,\varepsilon)}\right)=\frac{1-\omega^2}{(1-\varepsilon)S(\omega,\varepsilon)}(P_l^m)'\left(\frac{(1-\varepsilon)\omega}{S(\omega,\varepsilon)}\right) \tag{15}$$

with $S(\omega,\varepsilon)=\sqrt{1-\omega^2+(1-\varepsilon)^2\omega^2}$. For a sphere, equation (15) simplifies into

$$(1-\omega^2)\frac{\mathrm{d}P_l^m}{\mathrm{d}\omega}=mP_l^m \tag{16}$$

as is well-known [8]. As expected, when $m=0$ one may pass from equations (13–15) by changing ω^2 into $1-\omega^2$.

2.5 Some analytical solutions in a spherical shell

In the case of a spherical shell the above method cannot be applied for, on the inner shell, variables in the boundary conditions cannot be separated. However, a few analytical solutions exist and consist in purely non-axisymmetric toroidal modes. The eigenvalues are

$$\lambda_m=\mathrm{i}\sqrt{\frac{m}{m+1}} \tag{17}$$

and the spherical harmonics components are

$$\theta_m^m(r)=r^m \quad \text{and} \quad w_m^m(r)=\frac{\omega}{m}r^m$$

(see Sect. 3.1 for definitions).

Such solutions are related to the purely toroidal inertial modes found by Rieutord and Valdettaro [3]; for instance the $m=1$ mode is a solid-body rotation with frequency $1/\sqrt{2}$ which is the corresponding mode of the inertial "spin-over" mode.

3 Numerical solutions for modes in a spherical shell

As mentioned above, if the container is a spherical shell separation of variables cannot be achieved. We therefore have to resort to a complete numerical resolution the method of which is now presented.

In order to make the problem well-posed and avoid singularities, it is necessary to include viscosity and/or thermal diffusion. We therefore rewrite the equations of perturbations, associated with gravity modes as

$$\begin{cases}\lambda\mathbf{u}=-\boldsymbol{\nabla}p+\theta\mathbf{e}_z+E\Delta\mathbf{u}\\ \lambda\theta+u_z=K\Delta\theta\\ \boldsymbol{\nabla}\cdot\mathbf{u}=0\end{cases} \tag{18}$$

where we introduced two dimensionless numbers:

$$E=\frac{\nu}{NR^2} \qquad K=\frac{\kappa}{NR^2} \tag{19}$$

which measure the diffusion times with respect to the Brunt-Väisälä period. We note that E is the analogous of the Ekman number for inertial waves. λ is the complex eigenvalue of the mode. In the following we shall often use the notation $\lambda=\tau+\mathrm{i}\omega$, with $\tau<0$ being the damping rate and ω the pulsation of the mode.

Equations (18) must be completed by boundary conditions on the velocity and temperature. Although no-slip ($\mathbf{u}=\mathbf{0}$) boundary conditions are the realistic ones for the velocity field, we shall rather use stress-free boundary conditions, *i.e.*

$$u_\mathrm{r}=0 \quad \text{and} \quad \mathbf{e}_\mathrm{r}\times[\sigma]\mathbf{e}_\mathrm{r}=\mathbf{0}$$

where $[\sigma]$ is the stress tensor. Such boundary conditions are indeed more convenient for showing the presence of internal shear layers which are otherwise dominated by boundary layers. For temperature fluctuations, we use the perfect insulator boundary conditions and set

$$\frac{\partial\theta}{\partial r}=0 \quad \text{on} \quad r=\eta \quad \text{and} \quad r=1.$$

3.1 Numerical method

For solving the system (18) we first project these equations on spherical harmonics. We therefore write

$$\mathbf{u}=\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}u_m^l\mathbf{R}_l^m+v_m^l\mathbf{S}_l^m+w_m^l\mathbf{T}_l^m,$$
$$\theta=\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\theta_m^lY_l^m$$

with

$$\mathbf{R}_l^m=Y_l^m\mathbf{e}_r,\quad \mathbf{S}_l^m=\boldsymbol{\nabla}Y_l^m,\quad \mathbf{T}_l^m=\boldsymbol{\nabla}\times\mathbf{R}_l^m$$

which are the vectorial spherical harmonics (see also [10]).

The projection of the curl of the momentum equation together with the temperature equation yields three equations for the radial functions of order l:

$$\begin{cases}E\Delta_lw_m^l-\lambda w_m^l+\dfrac{\mathrm{i}m}{l(l+1)}\theta_m^l=0\\[2ex] K\Delta_l\theta_m^l-\lambda\theta_m^l+\mathrm{i}mw_m^l=\\ \quad\dfrac{1}{l(l+1)}\left[\beta_l^{l-1}r^{l-1}\dfrac{\partial}{\partial r}\left(\dfrac{ru^{l-1}}{r^{l-1}}\right)\right.\\ \qquad\left.+\beta_l^{l+1}r^{-l-2}\dfrac{\partial}{\partial r}\left(r^{l+2}ru^{l+1}\right)\right]\\[2ex] E\Delta_l\Delta_l(ru_m^l)-\lambda\Delta_l(ru_m^l)=\\ \quad\beta_l^{l-1}r^{l-1}\dfrac{\partial}{\partial r}\left(\dfrac{\theta^{l-1}}{r^{l-1}}\right)\\ \qquad+\beta_l^{l+1}r^{-l-2}\dfrac{\partial}{\partial r}\left(r^{l+2}\theta^{l+1}\right)\end{cases} \tag{20}$$

where we noted:

$$\beta_l^{l+1} = l\alpha(l+1),$$
$$\beta_l^{l-1} = -(l+1)\alpha(l),$$
$$\alpha(l) = \sqrt{\frac{l^2-m^2}{4l^2-1}},$$
$$\Delta_l = \frac{1}{r}\frac{\partial^2}{\partial r^2}r - \frac{l(l+1)}{r^2}\,.$$

This set of equations shows the coupling between harmonic components of order $l-1$, l, $l+1$ due to the cylindrical geometry of the stratification. The set is infinite since the equations are not separable in spherical coordinates. We also note that modes with different m's are not coupled, neither those of different parity (*i.e.* modes with $l-m$ even or odd hereafter denoted m^+ and m^- respectively). This system is very similar to the one of inertial modes (see [3]) especially for axisymmetric modes. In this latter case, the temperature fluctuation plays a similar role as the toroidal velocity field (*i.e.* the $w_m^l \mathbf{T}_l^m$ component) of inertial modes provided that the Prandtl number is unity; however the coupling coefficients are slightly different.

We solve the system (20), after truncation at a given order $L_{\max}$, by using a collocation method on the Gauss-Lobatto grid. The system thus changes into a generalized eigenvalue problem which we solve using standard eigenvalue solvers (see [3]).

3.2 The eigenvalue spectrum

When regular eigensolutions exist at zero diffusivities, it may easily be seen from (A.2) that damping rate will decrease as the $\max(K, E)$ as neither the functions nor their gradients will depend on these parameters in the asymptotic regime of vanishing diffusivities. Therefore if E dominates over K (as is the case in experiments with salty water), we expect a linear decrease of τ with E. This is indeed the case when the container is a full sphere (or an axisymmetric ellipsoid).

When the container is a spherical shell, computations of the full spectrum at various values of E and K clearly show (see Fig. 1) that all the damping rates decrease more slowly than E: If we write $\tau \propto E^\alpha$, then $\alpha < 1$. This is of course not true for the few purely toroidal modes which possess a limit at $E = K = 0$ (note how these modes emerge from the "sea" of eigenvalues in Fig. 1).

3.3 The shape of the modes

Our current understanding of this situation is that most of the modes are featured by the propagation of characteristics in the shell. Depending on the frequency, the trajectory of characteristics may tend to a limit cycle or cover the whole domain. In the first case, it seems that the associated eigenmode is "focused" along the limit cycle and the region where the solution is non zero (the support of the function) looks like rays travelling in the shell along characteristics directions. This focusing is associated with the negative value of the Lyapunov exponent associated with periodic orbits and we conjecture that α is related to this exponent. We therefore expect that at zero viscosity all of the modes associated with periodic orbit will be singular in the sense that the velocity field diverges on the periodic orbit. We refer the reader to Rieutord and Valdettaro [3] and Dintrans *et al.* [4] for a discussion on singularities within such modes.

We show in Figure 2 a gravity mode in a spherical shell and its corresponding inertial mode. Although viscosity and thermal diffusion have been included, we see that the meridional distribution of kinetic energy or viscous dissipation are noticeably similar. The damping rate is slightly higher for the inertial mode; this is likely a consequence of the excitation of the toroidal velocity field.

4 Discussion

The above results clearly illustrate the very special nature of gravity or inertial modes of a fluid contained in a spherical shell. Such kind of modes may play an important role in mixing processes. Indeed, suppose we are interested in the transport of a scalar quantity which concentration is c. A passing wave will generate a perturbation δc which expresses as

$$\delta c = -\mathbf{v}\cdot\boldsymbol{\nabla} c_0/\lambda$$

where $\mathbf{v}$ is the velocity field of the wave and λ its complex frequency. c_0 is the concentration profile before perturbation.

Passing waves will produce a secular transport which is characterized by the current

$$\mathbf{J} = \Re(\mathbf{v}^*\delta c) = -\Re(\mathbf{v}^*\mathbf{v}\cdot\boldsymbol{\nabla} c_0/\lambda).$$

As an example, we assume that the initial concentration gradient is vertical and unity, *i.e.* $\boldsymbol{\nabla} c_0 = \mathbf{e}_z$. In this case,

$$J_z = -\frac{|v_z|^2}{|\lambda|^2}\tau$$

which means that the vertical transport will be much higher for singular mode than for regular ones since the damping rate τ is much larger for the former modes.

In Figure 3 we computed the horizontal and vertical components of the flux vector associated with the gravity mode shown in Figure 2. It appears rather clearly that the vertical flux is non-zero only along the periodic orbit.

Finally, we should note that turbulence might easily develop along shear layers associated with periodic orbits. It is shown in [3] that the width of these layers is controlled by two small parameters $E^{1/4}$ and $E^{1/3}$, the exact role of which is not known yet but, if the analysis of similar but steady layers [11] extends to these layers, then the scale of velocity gradient is $E^{1/4}$. The associated Reynolds number (Re) for such a shear layer is then

$$Re = Ro\,E^{-3/4}$$

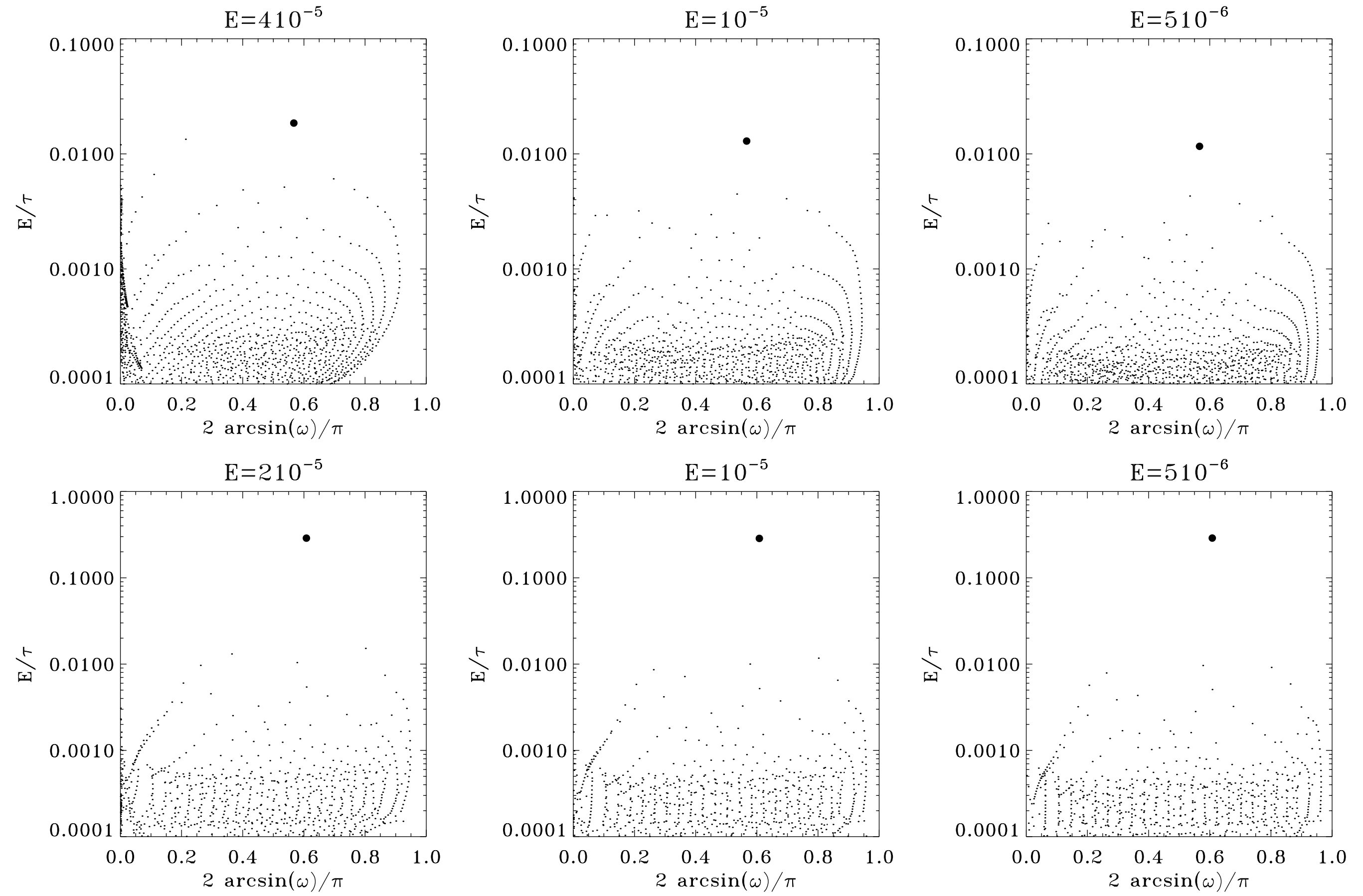

Fig. 1. Eigenvalues in the complex plane. The ordinate is in fact the life-time of the mode scaled by the E-number. The figures of the first row are modes of the 0^- symmetry while those of second row are of the 2^- symmetry. The black dot of each plot represents the position of the least-damped mode. Note how all the eigenvalues are moving to the lower part of the diagram except the least-damped mode with 2^- symmetry, which remains at a constant level since this mode has a regular limit at zero viscosity.

where Ro is the Rossby number. In geophysical flows, $E \sim 10^{-10}$ while $Ro \sim 0.1$; hence $Re \sim 10^7$ which is large enough for turbulence to develop during one oscillation.

To conclude, we may point out that an experimental investigation of the dynamics of stratified or rotating fluids may be pursued with either a stratified or a rotating fluid and the choice of it will depend on the quantities to be measured and the associated experimental constrains.

We have illustrated here, within the context of spherical geometry, the similarities and differences of the dynamics of the two fluids when low amplitude oscillations are considered. These differences may be summarized as follows: there is a one to one correspondence between axisymmetric inertial and gravity modes since the frequency of one type of mode may be obtained from the other type by just changing ω^2 into $1-\omega^2$. The corresponding expression of the pressure field is the same but the corresponding velocity field is different since the relation between these two fields are different for the two kinds of modes (see Eqs. (1, 5)). In the case of non-axisymmetric modes we have shown that one cannot make a one-to-one correspondence because of the boundary conditions met by the pressure field. However, from the fact that the trajectories of characteristics are independent of m, gravity modes and inertial modes will share the same patterns of characteristics provided one makes the change ω^2 to $1-\omega^2$. This implies that modes featured by shear layers may be found easily and in all cases by just computing the paths of characteristics.

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Appendix

We here gather some elementary properties of gravity modes. If we introduce the scalar product:

$$\langle f, g\rangle = \int_{(V)} f^* g \mathrm{d}V \quad \text{and} \quad \|f\| = \sqrt{\langle f, f\rangle}.$$

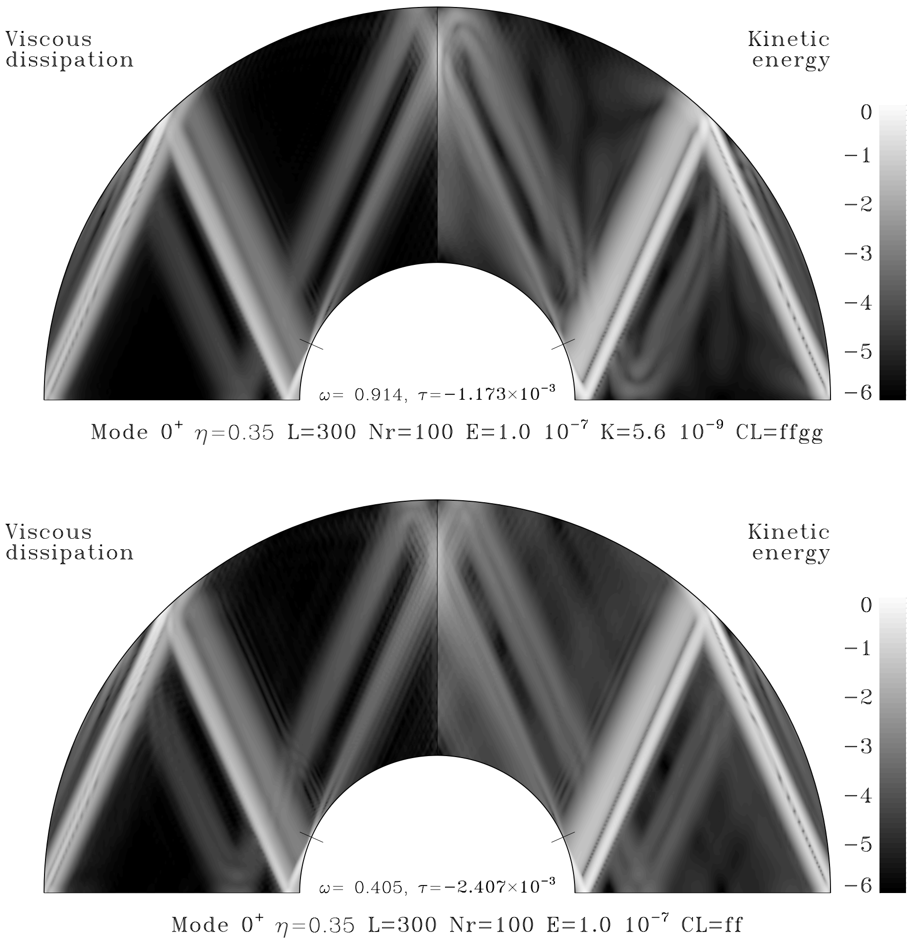

Fig. 2. A singular gravity mode in a spherical shell (above) and its corresponding inertial mode (below). The color-scale is (base 10) logarithmic.

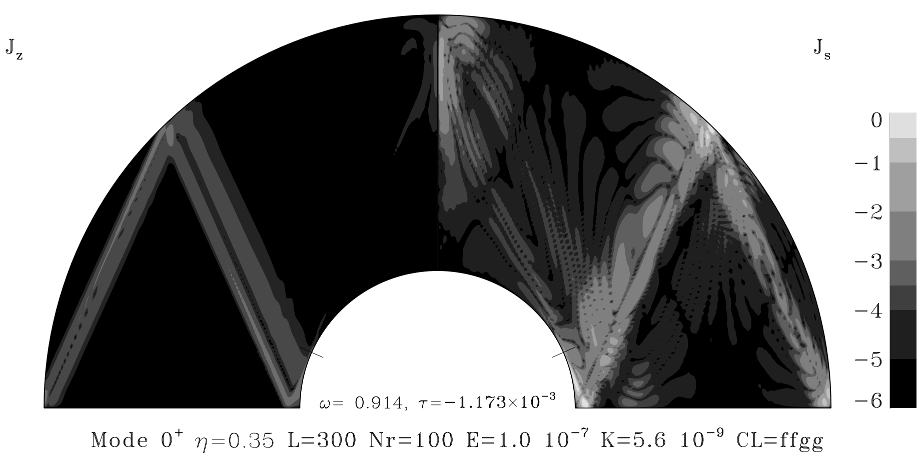

Fig. 3. Horizontal (J_s) and vertical (J_z) components of the flux vector, of a passive scalar resulting from the excitation of the mode drawn in Figure 2. The color-scale is (base 10) logarithmic.

The equation of motions yield

$$\lambda\|\mathbf{u}\|^2 = \langle u_z^*, \theta\rangle + E\langle \mathbf{u}^*, \Delta\mathbf{u}\rangle \quad \text{(A.1)}$$

$$\lambda\|\theta\|^2 + \langle \theta^*, u_z\rangle = K\langle \theta^*, \Delta\theta\rangle$$

from which we derive

$$\lambda\|\mathbf{u}\|^2 + \lambda^*\|\theta\|^2 = E\langle \mathbf{u}^*, \Delta\mathbf{u}\rangle + K\langle \theta, \Delta\theta^*\rangle.$$

Using boundary conditions we may rewrite this equation as

$$\lambda\|\mathbf{u}\|^2 + \lambda^*\|\theta\|^2 = -E\|\sigma\|^2 - K\|\nabla\theta\|^2$$

where σ is the rate-of-strain tensor. The real part of this expression yields the damping rate τ,

$$\tau = -\frac{E\|\sigma\|^2 + K\|\nabla\theta\|^2}{\|\mathbf{u}\|^2 + \|\theta\|^2} \quad \text{(A.2)}$$

while the imaginary part yields the equality

$$\|\mathbf{u}\|^2 = \|\theta\|^2 \quad \text{(A.3)}$$

for oscillatory modes (*i.e.* with $\omega \neq 0$). This relation shows the equipartition between kinetic and potential energy.

It is well-known that for a non-dissipative system the spectrum of gravity modes is bounded by the Brunt-Väisälä frequency (*i.e.* $\omega \leq N$). This result is also true for the dissipative system actually as may be seen directly from equation (A.1). Indeed, taking the imaginary part of this equation, one has

$$|\omega|\,\|\mathbf{u}\|^2 = |\langle u_z^*, \theta\rangle| \leq \|\mathbf{u}\|\,\|\theta\| \leq \|\mathbf{u}\|^2$$

using Cauchy-Schwartz inequalities. Thus $|\omega| \leq 1$ as expected.

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